General Methods in Proof Theory for Modal Logic – Lecture 2
Limits of the Sequent Framework

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Outline

Reminder: Modal Logics

The formulae of modal logic are given by ($\mathcal{V}$ is a set of variables):

$$\mathcal{F} \ ::= \ \mathcal{V} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \to \mathcal{F} \mid \neg \mathcal{F} \mid \Box \mathcal{F}$$

with $\Diamond A$ abbreviating the formula $\neg \Box \neg A$.

A Kripke frame consists of a set W of worlds and an accessibility relation $R \subseteq W \times W$.

A Kripke model is a Kripke frame with a valuation $V : \mathcal{V} \to \mathcal{P}(W)$.

Truth at a world w in a model $\mathfrak{M}$ is defined via:

$$\begin{aligned} \mathfrak{M}, w \Vdash p \ &\text{ iff } \ w \in V(p) \\ \mathfrak{M}, w \Vdash \Box A \ &\text{ iff } \ \forall v \in W : wRv \Rightarrow \mathfrak{M}, v \Vdash A \\ \mathfrak{M}, w \Vdash \Diamond A \ &\text{ iff } \ \exists v \in W : wRv \ \& \ \mathfrak{M}, v \Vdash A \end{aligned}$$

Modal Logic S5

Definition

Modal logic S5 is the logic given by the class of Kripke frames with universal accessibility relation, i.e., frames (W, R) with:

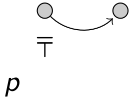
$$\forall x, y \in W : \; xRy \; .$$

Thus S5-theorems are those modal formulae which are true in every world of every Kripke model with universal accessibility relation.

Modal Logic S5

Example

The formulae $p \to \Box\Diamond p$ are theorems of S5:



Modal Logic S5

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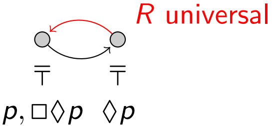

Modal Logic S5

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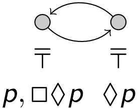

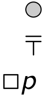

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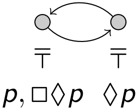

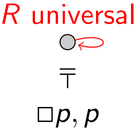

Modal Logic S5

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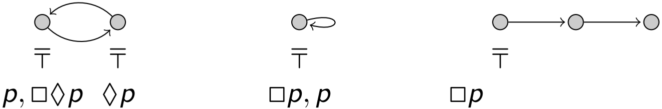

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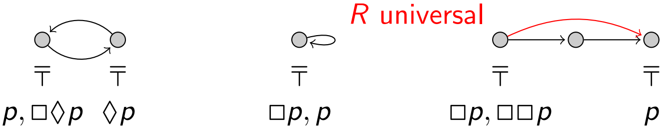

Modal Logic S5

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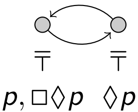

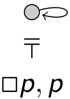

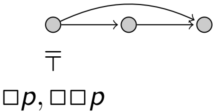

Hilbert-style Definition: S5 is given by closing the axioms

$$\Box(p \to q) \to (\Box p \to \Box q) \qquad p \to \Box\Diamond p \qquad \Box p \to p \qquad \Box p \to \Box\Box p$$

and propositional axioms under uniform substitution and the rules

$$\frac{A \quad A \to B}{B} \text{ modus ponens, MP} \qquad \frac{A}{\Box A} \text{ necessitation, nec}$$

A Sequent Calculus for S5

Definition (Takano 1992)

The sequent calculus sS5 contains the standard propositional rules and

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{T} \qquad \frac{\Box\Gamma \vdash A, \Box\Delta}{\Box\Gamma \vdash \Box A, \Box\Delta}\ 45$$

Theorem

sS5 *is sound and complete (with* cut*) for* S5*.*

Proof.

Derive axioms and rules of the Hilbert-system. E.g., for $p \to \Box\Diamond p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\neg p \vdash \Box\neg p}\ \text{init}}{\vdash \neg\Box\neg p, \Box\neg p}\ \neg_L}{\vdash \Box\neg\Box\neg p, \Box\neg p}\ 45 \qquad \dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{init}}{\neg p, p \vdash}\ \neg_L}{\Box\neg p, p \vdash}\ \mathsf{T}}{\dfrac{p \vdash \Box\neg\Box\neg p}{\vdash p \to \Box\Diamond p}\ \to_R}\ \text{cut}$$

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Theorem

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Proof.

E.g. the modus ponens rule $\dfrac{A \quad A \to B}{B}$ is simulated by:

$$\dfrac{\vdash A \qquad \dfrac{\vdash A \to B \qquad \dfrac{A, B \vdash B \quad A \vdash A, B}{A, A \to B \vdash B}\ \to_L}{A \vdash B}\ \mathsf{cut}}{\vdash B}\ \mathsf{cut}$$

$\Box$

What about cut-free completeness?

Our standard proof of cut elimination fails:

$$
\dfrac{\dfrac{\dfrac{\vdots}{\vdash \neg\Box\neg A, \Box\neg A}}{\vdash \Box\neg\Box\neg A, \Box\neg A}\,45 \qquad \dfrac{\dfrac{\vdots}{\neg A, A \vdash}}{\Box\neg A, A \vdash}\,\mathsf{T}}{A \vdash \Box\neg\Box\neg A}\,\text{cut}
$$

would need to reduce to:

$$
\dfrac{\dfrac{\dfrac{\vdots}{\vdash \neg\Box\neg A, \Box\neg A} \qquad \dfrac{\dfrac{\vdots}{\neg A, A \vdash}}{\Box\neg A, A \vdash}\,\mathsf{T}}{A \vdash \neg\Box\neg A}\,\text{cut}}{}\,??
$$

But we can't apply rule 45 anymore since A is not boxed!

What about cut-free completeness?

But could there be a different derivation?

No! In fact we have:

Theorem

The sequent $p \vdash \Box\Diamond p$ is not cut-free derivable in sS5*.*

Proof.

The only rules that can be applied in a cut-free derivation ending in $p \vdash \Box\Diamond p$ are weakening and contraction, possibly followed by 45. Hence, such a derivation can only contain sequents of one of the forms

$$p^{m} \vdash \Box\neg\Box\neg p^{n}$$

$$\Box\neg p^{m}, \neg p^{n} \vdash \Box\neg\Box\neg p^{k}, \neg\Box\neg p^{j}, p^{\ell}$$

with $m, n, k, \ell, j \geq 0$ and $A^{i} = \underbrace{A, \ldots, A}_{i\text{-times}}$. Thus it cannot contain an initial sequent. $\square$

How to show that a logic does not have a cut-free sequent calculus?

Is there a cut-free sequent calculus for S5?

Trivial answer: Of course!
Take the rules $\{ \frac{}{\vdash A} \mid A \text{ valid in S5}\}$.

Non-trivial answer: That depends on the shape of the rules!

General method for showing certain rule shapes cannot capture a semantically given modal logic even with cut:

- translate the rules into Hilbert-axioms of specific form
- connect Hilbert-style axiomatisability with frame definability
- show that the translations of the rules cannot define the frames for the logic.

(The translation involves cut, so this shows a stronger statement.)

What Is a Rule?

Let us call a sequent rule modal if it has the shape:

$$\frac{\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta}$$

where (writing $\Gamma^\Box$ for the restriction of Γ to modal formulae)

- $\Sigma_i \subseteq \Sigma,\ \Pi_i \subseteq \Pi$
- Γ_i is one of $\emptyset, \Gamma, \Gamma^\Box$
- Δ_i is one of $\emptyset, \Delta, \Delta^\Box$

Example

$$\frac{\Sigma \vdash A}{\Gamma, \Box\Sigma \vdash \Box A, \Delta}\ \mathsf{K} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{T} \quad \frac{\Gamma^\Box, \Sigma \vdash A}{\Gamma, \Box\Sigma \vdash \Box A, \Delta}\ 4 \quad \frac{\Gamma^\Box \vdash A, \Delta^\Box}{\Gamma \vdash \Box A, \Delta}\ 45$$

are all modal rules (and equivalent to the rules considered earlier).

What Is a Rule?

Let us call a sequent rule modal if it has the shape:

$$\frac{\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_1 \quad \ldots \quad \Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta}$$

where (writing $\Gamma^{\Box}$ for the restriction of Γ to modal formulae)

- $\Sigma_i \subseteq \Sigma,\ \Pi_i \subseteq \Pi$
- Γ_i is one of $\varnothing, \Gamma, \Gamma^{\Box}$
- Δ_i is one of $\varnothing, \Delta, \Delta^{\Box}$

Example

$$\frac{\Gamma^{\Box}, \Sigma, \Box A \vdash A}{\Gamma, \Box\Sigma \vdash \Box A, \Delta}\ \mathrm{GLR}$$

is not a modal rule (because the $\Box A$ changes sides).

Mixed-cut-closed Rule Sets

sS5 has modal rules in this sense, so we need something more.

Definition

A set of modal rules is mixed-cut-closed if principal-context cuts can be permuted up in the context.

Example

The set with modal rule $\dfrac{\Gamma^{\Box}, \Sigma \vdash A}{\Gamma \Box \Sigma \vdash \Box A, \Delta}\; 4$ is mixed-cut-closed: E.g.:

$$\dfrac{\dfrac{\Gamma^{\Box}, \Sigma \vdash A}{\Gamma, \Box\Sigma \vdash \Box A, \Delta}\;4 \qquad \dfrac{\Box A, \Omega^{\Box}, \Theta \vdash B}{\Box A, \Omega, \Box\Theta \vdash \Box B, \Xi}\;4}{\Gamma, \Box\Sigma, \Omega, \Box\Theta \vdash \Delta, \Box B, \Xi}\;\text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma^{\Box}, \Sigma \vdash A}{\Gamma^{\Box}, \Box\Sigma \vdash \Box A, \Delta}\;4 \qquad \Box A, \Omega^{\Box}, \Theta \vdash B}{\Gamma^{\Box}, \Sigma, \Omega^{\Box}, \Theta \vdash B}\;\text{cut}}{\Gamma, \Box\Sigma, \Omega, \Box\Theta \vdash \Delta, \Box B, \Xi}\;4$$

Mixed-cut-closed Rule Sets

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Definition

A set of modal rules is mixed-cut-closed if principal-context cuts can be permuted up in the context.

Example

The set sS5 is not mixed-cut-closed: the principal-context cut

$$\dfrac{\dfrac{\Gamma^\Box \vdash B, \Delta^\Box, \Box A}{\Gamma \vdash \Box B, \Delta, \Box A}\,45 \qquad \dfrac{\Sigma, A \vdash \Pi}{\Sigma, \Box A \vdash \Pi}\,T}{\Gamma, \Sigma \vdash \Box B, \Delta, \Pi}\,\text{cut}$$

cannot be permuted up in the context since Σ, Π are not boxed (see above).

Mixed-cut-closed Rule Sets Are Nice.

Lemma

If $\mathcal{R}$ is a mixed-cut-closed rule set for S5, then the contexts in all the premisses of the modal rules have one of the forms

$$\vdash \qquad \textit{or} \qquad \Gamma \vdash \Delta \qquad \textit{or} \qquad \Gamma^{\Box} \vdash \quad .$$

Idea of proof.

Show that every such rule set for S5 must include a rule similar to

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{T}$$

Use this rule and mixed-cut-closure to replace contexts $\Gamma^{\Box} \vdash \Delta^{\Box}$ with $\Gamma \vdash \Delta$. □

Step 1: Strategy for Translating Rules to Axioms

- We consider all the representative instances of a modal rule

$$\frac{\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta}$$

  i.e., instances of the modal rule where
  - Σ, Π consists of variables only
  - Γ, Δ consists of variables and boxed variables only
  - every variable occurs at most once in $\Gamma, \Delta, \Sigma, \Pi$.
- Premisses and conclusion of these are turned into the formulae

$$\begin{aligned} \text{prem} &= \textstyle\bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \bigvee \Delta_i) \\ \text{conc} &= \textstyle\bigwedge \Gamma \wedge \bigwedge \Box\Sigma \to \bigvee \Box\Pi \vee \bigvee \Delta \end{aligned}$$

- The information of the premisses is captured in a substitution σ_{prem} and injected into the conclusion by taking $\text{conc}\,\sigma_{\text{prem}}$

Constructing The Substitution σ_{prem}

We assume that our rule set includes the Monotonicity Rule

$$\frac{A \vdash B}{\Gamma, \Box A \vdash \Box B, \Delta}\ \text{Mon}$$

Definition (Adapted from [Ghilardi:'99])

A formula A is (S5-)projective via a substitution $\sigma : \mathcal{V} \to \mathcal{F}$ of variables by formulae if:

1. $\vdash A\,\sigma$ is derivable in GcutMon
2. for every $B \in \mathcal{F}$ the rule $\dfrac{\vdash A}{\vdash B \leftrightarrow B\sigma}$ is derivable in GcutMon.

Remark

For 2 it is enough to show for every $p \in \mathcal{V}$ derivability of the rule

$$\frac{\vdash A}{\vdash p \leftrightarrow p\sigma}\,.$$

Constructing The Substitution σ_{prem}

Lemma

The formula prem $= \bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \vee \bigvee \Delta_i)$ *is projective via*

$$\sigma_{\text{prem}}(p) = \begin{cases} \text{prem} \wedge p, & p \in \Sigma \\ \text{prem} \to p, & p \in \Pi \\ p, & \textit{otherwise} \end{cases}$$

Proof.

- To see that $\vdash_{\text{GcutMon}} \vdash \text{prem}\,\sigma_{\text{prem}}$:

For every clause $(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \vee \bigvee \Delta_i)$ of prem we have:

$$\begin{aligned} &(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \vee \bigvee \Delta_i)\sigma_{\text{prem}} \\ &\equiv \bigwedge \Gamma_i \wedge \bigwedge \Sigma_i\sigma_{\text{prem}} \to \bigvee \Pi_i\sigma_{\text{prem}} \vee \bigvee \Delta_i \\ &\equiv \bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \wedge \text{prem} \to \bigvee \Pi_i \vee \bigvee \Delta_i \end{aligned}$$

Since $(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \vee \bigvee \Delta_i)$ is a clause in prem this is derivable.

Constructing The Substitution σ_{prem}

Lemma

The formula $\text{prem} = \bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \vee \bigvee \Delta_i)$ *is projective via*

$$\sigma_{\text{prem}}(p) = \begin{cases} \text{prem} \wedge p, & p \in \Sigma \\ \text{prem} \to p, & p \in \Pi \\ p, & \textit{otherwise} \end{cases}$$

Proof.

- To see that $\dfrac{\vdash \text{prem}}{\vdash p \leftrightarrow p\sigma_{\text{prem}}}$ is derivable is straightforward:

E.g., for $p \in \Pi$:

$$\dfrac{\dfrac{}{p \vdash \text{prem} \to p}\text{prop} \qquad \dfrac{\vdash \text{prem} \qquad \dfrac{}{\text{prem}, \text{prem} \to p \vdash p}\text{prop}}{\text{prem} \to p \vdash p}\text{cut}}{\vdash p\sigma_{\text{prem}} \leftrightarrow p}\text{prop}$$

□

Theorem

A modal rule

$$\frac{\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta}\ R$$

is interderivable over GcutMon *with the axioms* $\mathsf{conc}\,\sigma_{\mathsf{prem}}$ *obtained from its representative instances.*

Proof.

Derive the rule from the axiom using:

$$\dfrac{\dfrac{\vdash \mathsf{conc}\,\sigma_{\mathsf{prem}} \qquad \dfrac{\dfrac{\dfrac{\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n}{\vdash \mathsf{prem}}\ \mathsf{prop}}{\vdash \mathsf{conc} \leftrightarrow \mathsf{conc}\,\sigma_{\mathsf{prem}}}\ \mathsf{projectivity}}{\mathsf{conc}\,\sigma_{\mathsf{prem}} \vdash \mathsf{conc}}\ \mathsf{prop}}{\vdash \mathsf{conc}}\ \mathsf{cut}}{\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta}\ \mathsf{prop}$$

Theorem

A modal rule

$$\frac{\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta}\ R$$

is interderivable over GcutMon *with the axioms* $\mathsf{conc}\,\sigma_{\mathsf{prem}}$ *obtained from its representative instances.*

Proof.

Derive the axiom from the rule by:

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{prem}\,\sigma_{\mathsf{prem}}}\ \text{projectivity}}{(\Gamma_1, \Sigma_1 \vdash \Pi_1, \Delta_i)\,\sigma_{\mathsf{prem}}}\ \text{prop} \quad \dots \quad \dfrac{\dfrac{}{\vdash \mathsf{prem}\,\sigma_{\mathsf{prem}}}\ \text{projectivity}}{(\Gamma_n, \Sigma_n \vdash \Pi_n, \Delta_n)\,\sigma_{\mathsf{prem}}}\ \text{prop}}{\dfrac{(\Gamma, \Box\Sigma \vdash \Box\Pi, \Delta)\,\sigma_{\mathsf{prem}}}{\vdash \mathsf{conc}\,\sigma_{\mathsf{prem}}}\ \text{prop}}\ R$$

□

Example

The rule $\dfrac{\Gamma^{\Box} \vdash A, \Delta^{\Box}}{\Gamma \vdash \Box A, \Delta}$ 45 has representative instances

$$\frac{\Box p_1, \ldots, \Box p_n \vdash q, \Box r_1, \ldots, \Box r_k}{\Box p_1, \ldots, \Box p_n \vdash \Box q, \Box r_1, \ldots, \Box r_k}$$

The formulae and substitution are

$$\text{prem} = \bigwedge_{i=1}^{n} \Box p_i \to q \vee \bigvee_{j=1}^{k} \Box r_j \qquad \text{conc} = \bigwedge_{i=1}^{n} \Box p_i \to \Box q \vee \bigvee_{j=1}^{k} \Box r_j$$

$$\sigma_{\text{prem}}(q) = \text{prem} \to q \qquad \sigma_{\text{prem}}(s) = s \text{ for } s \neq q$$

E.g., for $n = 1$ and $k = 1$ the corresponding axiom is:

$$\text{conc}\,\sigma_{\text{prem}} = \Box p_1 \to \Box((\Box p_1 \to q \vee \Box r_1) \to q) \vee \Box r_1$$

Instantiating q with $\bot$ we have the instance

$$\Box p_1 \to \Box(\Box p_1 \wedge \neg \Box r_1) \vee \Box r_1 \quad \equiv \quad (\Box p_1 \to \Box \Box p_1) \wedge (\Diamond \Box r_1 \to \Box r_1)$$

Step 2: What Do The Axioms Look Like?

An exemplary representative instance of a modal rule from a mixed-cut-closed rule set has the form

$$\frac{\Sigma_1 \vdash \Pi_1 \qquad p, \Box q, \Sigma_2 \vdash \Pi_2, r \qquad \Box q, \Sigma_3 \vdash \Pi_3}{p, \Box q, \Box\Sigma \vdash \Box\Pi, r}$$

The formula prem is

$$(\bigwedge \Sigma_1 \to \bigvee \Pi_1) \wedge (p, \Box q \wedge \bigwedge \Sigma_2 \to \bigvee \Pi_2 \vee r) \wedge (\Box q \wedge \bigwedge \Sigma_3 \to \bigwedge \Pi_3)$$

and the axiom is

$$A_{\mathsf{S5}} \quad = \quad p \wedge \Box q \wedge \bigwedge_{s \in \Sigma} \Box(\text{prem} \wedge s) \to \bigvee_{t \in \Pi} \Box(\text{prem} \to t) \vee r$$

Step 3: Such axioms cannot define S5.

Lemma

If $\neg A_{S5}$ is satisfiable in one of the frames $\mathfrak{F} = (\mathbb{N}, \mathbb{N} \times \mathbb{N})$ and $\mathfrak{F}' = (\mathbb{N}, \leq)$, then it is also satisfiable in the other.

Proof.

$$\neg A_{S5} \quad \equiv \quad p \wedge \Box q \wedge \bigwedge_{s \in \Sigma} \Box(\text{prem} \wedge s) \wedge \bigwedge_{t \in \Pi} \Diamond(\text{prem} \wedge \neg t) \wedge \neg t$$

E.g., if $\mathfrak{F}', V', 1 \Vdash \neg A$ for a valuation V', then $\mathfrak{F}, V, 0 \Vdash \neg A$ with

$$V(n) := V'(n+1)$$

(The only boxed formula in prem is $\Box q$!) $\Box$

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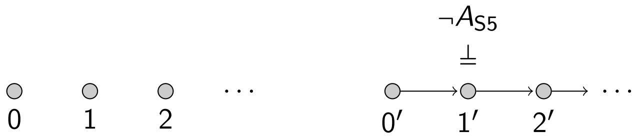

Proof.

$$\neg A_{\mathsf{S5}} \quad \equiv \quad p \wedge \Box q \wedge \textstyle\bigwedge_{s \in \Sigma} \Box(\mathrm{prem} \wedge s) \wedge \bigwedge_{t \in \Pi} \Diamond(\mathrm{prem} \wedge \neg t) \wedge \neg t$$

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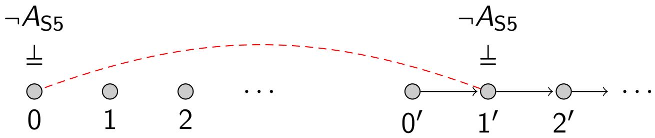

Proof.

$$\neg A_{S5} \quad \equiv \quad p \wedge \Box q \wedge \bigwedge_{s \in \Sigma} \Box(\text{prem} \wedge s) \wedge \bigwedge_{t \in \Pi} \Diamond(\text{prem} \wedge \neg t) \wedge \neg t$$

E.g., if $\mathfrak{F}', V', 1 \Vdash \neg A$ for a valuation V', then $\mathfrak{F}, V, 0 \Vdash \neg A$ with

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(The only boxed formula in prem is $\Box q$!) $\Box$

No Mixed-cut-closed Rule Sets for S5

Theorem

No sequent calculus with mixed-cut-closed propositional and modal rules is sound and complete for S5 *(even with* cut*).*

Proof.

- The translations of such rules would have a shape like A_{S5} above.
- By the Lemma, such axioms are valid in the S5-frame $(\mathbb{N}, \mathbb{N} \times \mathbb{N})$ iff they are valid in $(\mathbb{N}, \leq)$
- So all axioms (and hence: theorems) of S5 would be valid in $(\mathbb{N}, \leq)$ – but e.g. $p \to \Box\Diamond p$ is not.

$\square$

Other Limitative Results Using this Method

Theorem

No mixed-cut closed sequent calculus with modal rules captures:

- *provability logic* GL
- *modal logic of symmetry* KB*:* $xRy \Rightarrow yRx$
- *modal logic of 2-transitivity:* $xRy \& yRz \& zRw \Rightarrow \exists v.xRv \& vRw$

Definition

- A shallow rule has no modal restriction on the context formulae.
- A one-step rule has no context formulae.

Theorem

- *No calculus with only shallow rules captures* K4
- *No calculus with only one-step rules captures* KT

Can we extend the sequent framework to obtain a cut-free sequent-style calculus for logics like S5?

Hypersequent Calculi

Hypersequents

General idea

Consider several sequents in parallel, allowing for interaction!

Definition

A hypersequent is a multiset $\mathcal{G}$ of sequents, written as

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n \, .$$

The sequents $\Gamma_i \vdash \Delta_i$ are called the components of $\mathcal{G}$.

Hypersequent calculi for S5 were independently introduced in

[Mints:’74], [Pottinger:’83], [Avron:’96]

Hypersequents were also used to provide cut-free calculi for many other logics including modal, substructural and intermediate logics.

Hypersequents for S5

The (S5-)interpretation of $\mathcal{G} = \Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$ is

$$\iota(\mathcal{G}) \quad := \quad \Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \cdots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n)$$

This interpretation suggests the external structural rules

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \vdash \Delta}\ \mathsf{EW} \qquad\qquad \frac{\mathcal{G} \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{\mathcal{G} \mid \Gamma \vdash \Delta}\ \mathsf{EC}$$

Hypersequent Rules for S5

The calculus hsS5 for S5 contains the modal rules

$$\frac{\mathcal{G} \mid \Gamma \vdash \Delta \mid \ \vdash A}{\mathcal{G} \mid \Gamma \vdash \Delta, \Box A}\ \Box_R \qquad \frac{\mathcal{G} \mid \Gamma \vdash \Delta \mid \Sigma, A \vdash \Pi}{\mathcal{G} \mid \Gamma, \Box A \vdash \Delta \mid \Sigma \vdash \Pi}\ \Box_L \qquad \frac{\mathcal{G} \mid \Gamma, A \vdash \Delta}{\mathcal{G} \mid \Gamma, \Box A \vdash \Delta}\ \mathsf{T}$$

the standard propositional rules in every component and the external structural rules [Restall:'07].

Example

The derivations of $p \vdash \Box\Diamond p$ and $\Box p \vdash \Box\Box p$ are as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p \mid \vdash}\ \text{init}}{p, \neg p \vdash \mid \vdash}\ \neg_L}{p \vdash \mid \Box\neg p \vdash}\ \Box_L}{p \vdash \mid \vdash \neg\Box\neg p}\ \neg_R}{p \vdash \Box\neg\Box\neg p}\ \Box_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mid \vdash \mid p \vdash p}\ \text{init}}{\Box p \vdash \mid \vdash \mid \vdash p}\ \Box_L}{\Box p \vdash \mid \vdash \Box;}\ \Box_R}{\Box p \vdash \Box\Box p}\ \Box_R$$

Soundness of hsS5

Theorem

The rules of hsS5 *preserve validity under the* S5*-interpretation.*

Proof.

E.g., for $\dfrac{\mathcal{G} \mid \Gamma \vdash \Delta \mid \Sigma, A \vdash \Pi}{\mathcal{G} \mid \Gamma, \Box A \vdash \Delta \mid \Sigma \vdash \Pi}\ \Box_L$:

If $\mathfrak{M}, w \Vdash \neg\iota(\mathcal{G}) \wedge \Diamond(\bigwedge\Gamma \wedge \Box A \wedge \neg\bigvee\Delta) \wedge \Diamond(\bigwedge\Sigma \wedge \neg\bigvee\Pi)$ we have:

$$\begin{tikzcd}
\neg\iota(\mathcal{G}) \dashv\mid w \arrow[r] \arrow[rr, bend left] & x \arrow[d, phantom, "\Vdash"] & y \Vdash \bigwedge\Sigma, \quad \neg\bigvee\Pi \\
& \bigwedge\Gamma, \Box A, \neg\bigvee\Delta &
\end{tikzcd}$$

Soundness of hsS5

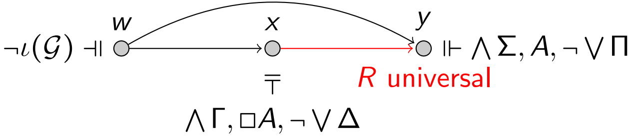
Theorem

The rules of hsS5 *preserve validity under the* S5*-interpretation.*

Proof.

E.g., for $\dfrac{\mathcal{G} \mid \Gamma \vdash \Delta \mid \Sigma, A \vdash \Pi}{\mathcal{G} \mid \Gamma, \Box A \vdash \Delta \mid \Sigma \vdash \Pi}\ \Box_L$:

If $\mathfrak{M}, w \Vdash \neg\iota(\mathcal{G}) \wedge \Diamond(\bigwedge \Gamma \wedge \Box A \wedge \neg \bigvee \Delta) \wedge \Diamond(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$ we have:



So $\mathfrak{M}, w \Vdash \neg\iota(\mathcal{G} \mid \Gamma \vdash \Delta \mid \Sigma, A \vdash \Pi)$. $\Box$

Soundness of hsS5

Theorem

The rules of hsS5 *preserve validity under the* S5*-interpretation.*

Corollary

If $\vdash A$ *is derivable in* hsS5, *then* A *is valid in* S5.

Proof.

By induction on the depth of the derivation, and using that the rule

$$\frac{\Box A}{A}$$

is admissible in S5. $\Box$

Completeness of hsS5

We first show completeness with the hypersequent cut rule

$$\frac{\mathcal{G} \mid \Gamma \vdash \Delta, A \qquad \mathcal{H} \mid A, \Sigma \vdash \Pi}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \vdash \Delta, \Pi}\ \text{hcut}$$

Theorem

If A is S5*-valid, then* $\vdash A$ *is derivable in* hsS5 *with* hcut*.*

Proof.

Derive the axioms of S5 and simulate the rule of modus ponens by:

$$\dfrac{\begin{array}{c}\vdots\\ \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash A \to B\end{array} \qquad \dfrac{\dfrac{}{B, A \vdash B}\ \text{init} \qquad \dfrac{}{A \vdash A, B}\ \text{init}}{A \to B, A \vdash B}\ {\to_L}}{\vdash A \to B}\ \text{hcut}}{\vdash B}\ \text{hcut}$$

$\square$

Hypersequent Cut Elimination - Complications

Cut elimination for hypersequents is complicated by the external structural rules, in particular by the rule of external contraction:

E.g. we might have the situation

$$\dfrac{\mathcal{G} \mid \Gamma \vdash \Delta, A \qquad \dfrac{\mathcal{H} \mid A, \Sigma \vdash \Pi \mid A, \Sigma \vdash \Pi}{\mathcal{H} \mid A, \Sigma \vdash \Pi}\ \text{EC}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \vdash \Delta, \Pi}\ \text{hcut}$$

Permuting the cut upwards replaces it by two cuts of the same complexity:

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma \vdash \Delta, A \qquad \dfrac{\mathcal{G} \mid \Gamma \vdash \Delta, A \qquad \mathcal{H} \mid A, \Sigma \vdash \Pi \mid A, \Sigma \vdash \Pi}{\mathcal{G} \mid \mathcal{H} \mid A, \Sigma \vdash \Pi \mid \Gamma, \Sigma \vdash \Delta, \Pi}\ \text{hcut}}{\mathcal{G} \mid \mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \vdash \Delta, \Pi \mid \Gamma, \Sigma \vdash \Delta, \Pi}\ \text{hcut}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \vdash \Delta, \Pi}\ \text{EC}$$

Cut Elimination for hsS5 - Outline

Several methods of cut elimination are possible.
Here we follow one which generalises rather well
[Ciabattoni:'10, L.:'14].

Strategy

- pick a top-most cut of maximal complexity
- shift up to the left until the cut formula is introduced ("Shift Left Lemma")
- shift up to the right until the cut formula is introduced ("Shift Right Lemma")
- reduce the complexity of the cut

Key Ingredient

Absorb contractions by considering a more general induction hypothesis, similar to a one-sided mix rule.

Cut Elimination for hsS5 - Shift Right Lemma

Definition

The cut rank of a derivation in hsS5hcut is the maximal complexity $|A|$ of a cut formula A in it.

Lemma (Shift Right Lemma)

If there are hsS5hcut-*derivations*

$$\begin{array}{c} \vdots\ \mathcal{D} \\ \mathcal{G} \mid \Gamma \vdash \Delta, A \end{array} \qquad \textit{and} \qquad \begin{array}{c} \vdots\ \mathcal{E} \\ \mathcal{H} \mid A^{k_1}, \Sigma_1 \vdash \Pi_1 \mid \ldots \mid A^{k_n}, \Sigma_n \vdash \Pi_n \end{array}$$

of cut rank $< |A|$ *with* A *principal in the last rule of* $\mathcal{D}$*, then there is a derivation of cut rank* $< |A|$ *of*

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \vdash \Delta, \Pi_1 \mid \ldots \mid \Gamma, \Sigma_n \vdash \Delta, \Pi_n \,.$$

Proof (Shift Right Lemma).

By induction on the depth of the derivation $\mathcal{E}$, distinguishing cases according to the last rule in $\mathcal{E}$. Some interesting cases:

- Last applied rule EC:

$$\dfrac{\vdots\ \mathcal{D}}{\mathcal{G} \mid \Gamma \vdash \Delta, A} \qquad \dfrac{\dfrac{\vdots\ \mathcal{E}'}{\mathcal{H} \mid A^{k_1}, \Sigma_1 \vdash \Pi_1 \mid \ldots \mid A^{k_n}, \Sigma_n \vdash \Pi_n \mid A^{k_n}, \Sigma_n \vdash \Pi_n}}{\mathcal{H} \mid A^{k_1}, \Sigma_1 \vdash \Pi_1 \mid \ldots \mid A^{k_n}, \Sigma_n \vdash \Pi_n}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}}{\mathcal{G} \mid \Gamma \vdash \Delta, A} \qquad \dfrac{\vdots\ \mathcal{E}'}{\mathcal{H} \mid A^{k_1}, \Sigma_1 \vdash \Pi_1 \mid \ldots \mid A^{k_n}, \Sigma_n \vdash \Pi_n \mid A^{k_n}, \Sigma_n \vdash \Pi_n}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \vdash \Delta, \Pi_1 \mid \ldots \mid \Gamma, \Sigma_n \vdash \Delta, \Pi_n \mid \Gamma, \Sigma_n \vdash \Delta, \Pi_n}\ IH}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \vdash \Delta, \Pi_1 \mid \ldots \mid \Gamma, \Sigma_n \vdash \Delta, \Pi_n}\ \mathsf{EC}$$

Proof (Shift Right Lemma).

By induction on the depth of the derivation $\mathcal{E}$, distinguishing cases according to the last rule in $\mathcal{E}$. Some interesting cases:

- $A = \Box B$ and last applied rule $\Box_L$ with $\Box B$ principal (omitting side hypersequents and showing only two components):

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \Gamma \vdash \Delta \mid\ \vdash B\end{array}}{\Gamma \vdash \Delta, \Box B}\ \Box_R \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{E}'\\ \Box B^{k_1-1}, \Sigma_1 \vdash \Pi_1 \mid B, \Box B^{k_2}, \Sigma_2 \vdash \Pi_2\end{array}}{\Box B^{k_1}, \Sigma_1 \vdash \Pi_1 \mid \Box B^{k_2}, \Sigma_2 \vdash \Pi_2}\ \Box_L$$

$$\rightsquigarrow$$

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \Gamma \vdash \Delta \mid\ \vdash B\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \Gamma \vdash \Delta \mid\ \vdash B\end{array}}{\Gamma \vdash \Delta, \Box B}\ \Box_R \qquad \begin{array}{c}\vdots\ \mathcal{E}'\\ \Box B^{k_1-1}, \Sigma_1 \vdash \Pi_1 \mid B, \Box B^{k_2}, \Sigma_2 \vdash \Pi_2\end{array}}{\Gamma, \Sigma_1 \vdash \Delta, \Pi_1 \mid B, \Gamma, \Sigma_2 \vdash \Delta, \Pi_2}\ IH}{\Gamma, \Sigma_1 \vdash \Delta, \Pi_1 \mid \Gamma, \Sigma_2 \vdash \Delta, \Pi_2}\ \text{hcut}, \text{W}, \text{EC}$$

Cut Elimination for hsS5 - Shift Left Lemma

Lemma (Shift Left Lemma)

If there are hsS5hcut-*derivations*

$$\dfrac{\vdots\ \mathcal{D}}{\mathcal{G} \mid \Gamma_1 \vdash \Delta_1, A^{k_1} \mid \ldots \mid \Gamma_n \vdash \Delta_n, A^{k_n}} \qquad \textit{and} \qquad \dfrac{\vdots\ \mathcal{E}}{\mathcal{H} \mid A, \Sigma \vdash \Pi}$$

of cut rank $< |A|$*, then there is a derivation of cut rank* $< |A|$ *of*

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma \vdash \Delta_1, \Pi \mid \ldots \mid \Gamma_n, \Sigma \vdash \Delta_n, \Pi\ .$$

Proof (Shift Left Lemma)

By induction on the depth of the derivation $\mathcal{D}$, distinguishing cases according to the last rule in $\mathcal{D}$. An interesting case:

- $A = \Box B$ and last applied rule $\Box_R$ with $\Box B$ principal (omitting side hypersequents and assuming only two components):

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}' \\ \Gamma_1 \vdash \Delta_1, \Box B^{k_1} \mid \Gamma_2 \vdash \Delta_2, \Box B^{k_2-1} \mid\ \vdash B\end{array}}{\Gamma_1 \vdash \Delta_1, \Box B^{k_1} \mid \Gamma_2 \vdash \Delta_2, \Box B^{k_2}}\ \Box_R \qquad \begin{array}{c}\vdots\ \mathcal{E} \\ \Box B, \Sigma \vdash \Pi\end{array}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}' \\ \Gamma_1 \vdash \Delta_1, \Box B^{k_1} \mid \Gamma_2 \vdash \Delta_2, \Box B^{k_2-1} \mid\ \vdash B\end{array} \quad \begin{array}{c}\vdots\ \mathcal{E} \\ \Box B, \Sigma \vdash \Pi\end{array}}{\Gamma_1, \Sigma \vdash \Delta_1, \Pi \mid \Gamma_2, \Sigma \vdash \Delta_2, \Pi \mid\ \vdash B}\ IH}{\Gamma_1, \Sigma \vdash \Delta_1, \Pi \mid \Gamma_2, \Sigma \vdash \Delta_2, \Pi, \Box B}\ \Box_R \qquad \begin{array}{c}\vdots\ \mathcal{E} \\ \Box B, \Sigma \vdash \Pi\end{array}}{\Gamma_1, \Sigma \vdash \Delta_1, \Pi \mid \Gamma_2, \Sigma \vdash \Delta_2, \Pi}\ SRL$$

Cut Elimination for hsS5 - Main Theorem

Theorem

Every derivation in hsS5hcut *can be converted into a derivation in* hsS5 *with the same conclusion.*

Proof.

By double induction on the cut rank r of the derivation and the number of cuts on formulae with complexity r. Topmost cuts of maximal complexity are eliminated using the Shift Left Lemma. $\square$

Corollary (Cut-free Completeness)

If A is S5*-valid, then $\vdash A$ is derivable in* hsS5*.*

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive , single-conclusion right , right-contraction closed , mixed-cut permuting , principal cut closed set of hypersequent rules with context restrictions has cut elimination.

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1], single-conclusion right , right-contraction closed , mixed-cut permuting , principal cut closed set of hypersequent rules with context restrictions has cut elimination.

[1] Cuts on context formulae permute up on the left

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1]*, single-conclusion right*[2]*, right-contraction closed , mixed-cut permuting , principal cut closed set of hypersequent rules with context restrictions has cut elimination.*

[1] Cuts on context formulae permute up on the left

[2] No formula is introduced on the right in two components

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1]*, single-conclusion right*[2]*, right-contraction closed*[3]*, mixed-cut permuting , principal cut closed set of hypersequent rules with context restrictions has cut elimination.*

[1] Cuts on context formulae permute up on the left

[2] No formula is introduced on the right in two components

[3] No formula is introduced on the right twice in a component

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1]*, single-conclusion right*[2]*, right-contraction closed*[3]*, mixed-cut permuting*[4]*, principal cut closed set of hypersequent rules with context restrictions has cut elimination.*

[1] Cuts on context formulae permute up on the left

[2] No formula is introduced on the right in two components

[3] No formula is introduced on the right twice in a component

[4] Principal-context cuts permute up on the right

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1], single-conclusion right[2], right-contraction closed[3], mixed-cut permuting[4], principal cut closed[5] set of hypersequent rules with context restrictions has cut elimination.

[1] Cuts on context formulae permute up on the left

[2] No formula is introduced on the right in two components

[3] No formula is introduced on the right twice in a component

[4] Principal-context cuts permute up on the right

[5] Principal-principal cuts can be reduced

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1], single-conclusion right[2], right-contraction closed[3], mixed-cut permuting[4], principal cut closed[5] set of hypersequent rules with context restrictions[6] has cut elimination.

1 Cuts on context formulae permute up on the left
2 No formula is introduced on the right in two components
3 No formula is introduced on the right twice in a component
4 Principal-context cuts permute up on the right
5 Principal-principal cuts can be reduced
6 Suitably defined.

General Cut Elimination

From the proof for S5 we can extract sufficient conditions for applicability of the Shift-Left-Shift-Right method:

Theorem

Every right-substitutive[1], single-conclusion right[2], right-contraction closed[3], mixed-cut permuting[4], principal cut closed[5] set of hypersequent rules with context restrictions[6] has cut elimination.

1. Cuts on context formulae permute up on the left
2. No formula is introduced on the right in two components
3. No formula is introduced on the right twice in a component
4. Principal-context cuts permute up on the right
5. Principal-principal cuts can be reduced
6. Suitably defined.

For the dirty details see [L.:'14].

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