General Methods in Proof Theory for Modal Logic – Lecture 4
Non-normal Modal Logics and Logical Rules

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TU Wien

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Outline

An IndoLogical Problem

Imagine...

- You are an indologist and study texts of the Mīmāṃsā school of Indian Philosophy, concerned with analysing prescriptions contained in the Vedas, the sacred texts of Hinduism.

यत्र तूत्पत्त्यादयो न विध्यन्तरसिद्धास् तत्र स्वयमेव
स्वसम्बन्धिनामुत्पत्त्यादिचतुष्टयं करोति

An IndoLogical Problem

Imagine...

- You are an indologist and study texts of the Mīmāṃsā school of Indian Philosophy, concerned with analysing prescriptions contained in the Vedas, the sacred texts of Hinduism.
- You happen to meet an established proof theorist.
- In a lively discussion the two of you come up with the idea to use proof-theoretic reasoning to analyse different Mīmāṃsā authors by
  - extracting their modes of reasoning into (modal) logics;
  - constructing cut-free calculi for these logics;
  - comparing the different authors' interpretations using the corresponding calculi.

An IndoLogical Problem

Imagine further...

- In long, laborious work the two of you have managed to extract several modal logics from the texts.

  (In fact, you even extracted several modal logics for each author and are not sure which ones are best.)

So the only thing left to do is to analyse the logics using their proof theory. However, for this you need cut-free calculi for these logics...

An IndoLogical Problem

Imagine that you, the indologist, have extracted the logics from the texts by interpreting principles like

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(I.e., "When, on the other hand, coming into existence [of something needed], etc., are not realised by another prescription, [the principal prescription] itself begets the four [stages] of coming into being, etc., [of the prescriptions] connected to itself.")

as Hilbert-style axioms, e.g. (with $\mathcal{O}$ for "ought to"):

$$\Box(A \to B) \to (\mathcal{O}A \to \mathcal{O}B)$$

An IndoLogical Problem

Moreover, imagine that unfortunately you have not found evidence that the Mīmāṃsā logics for the modality $\mathcal{O}$ have a Kripke semantics.

This means that:

- You cannot use calculi based on Kripke semantics (e.g. labelled sequent systems).
- Even if your logics had Kripke semantics, to construct e.g. labelled sequent systems you would need to convert Hilbert-axioms into frame conditions (which can be tricky / impossible).

This problem leads to the obvious question...

How to construct sequent calculi for non-normal modal logics from Hilbert-axioms?

Non-normal Modal Logics Axiomatically

Definition

Classical modal logic E is given Hilbert-style by closing axioms for propositional logic under the rules

$$\frac{A \quad A \to B}{B} \text{ modus ponens, MP} \qquad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B} \text{ congruence, Cg}$$

A classical modal logic is given by extending the Hilbert-system for E with further axioms.

Examples

The standard non-normal modal logics extend E with axioms from

(m) $\Box(A \wedge B) \to \Box A$ (c) $\Box A \wedge \Box B \to \Box(A \wedge B)$ (n) $\Box\top$

E.g., logic EC adds axiom (c), logic ECN adds (c), (n), etc. Logic EM is called monotone logic M. Note that MCN is modal logic K.

Non-normal Modal Logics Semantically

Definition

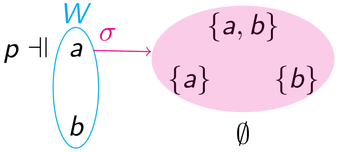
A neighbourhood frame consists of a set W of worlds and a neighbourhood function $\sigma : W \to \mathcal{P}(\mathcal{P}(W))$. A neighbourhood model adds a valuation $V : \mathcal{V} \to \mathcal{P}(W)$.

Modal formulae are evaluated in a model $\mathfrak{M}$ at a world w using:

$$\mathfrak{M}, w \Vdash \Box A \text{ iff } \{v \in W : \mathfrak{M}, v \Vdash A\} \in \sigma(w)$$

Example

Consider the neighbourhood model $\mathfrak{M} = (W, \sigma, V)$ given by:



- $\{v \in W : \mathfrak{M}, v \Vdash p\} \in \sigma(a)$, thus $\mathfrak{M}, a \Vdash \Box p$
- $\{v \in W : \mathfrak{M}, v \Vdash q\} \notin \sigma(a)$, thus $\mathfrak{M}, a \Vdash \neg\Box q$

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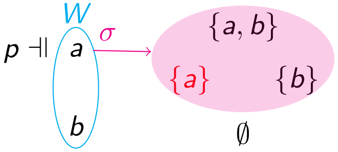

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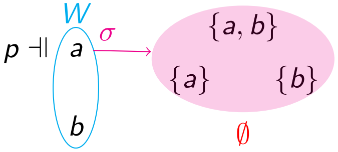

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Non-normal Modal Logics Semantically

Additional axioms often correspond to conditions on the neighbourhood frames, e.g.:

- (m) $\Box(A \wedge B) \to \Box A \quad \leftrightsquigarrow \quad A \in \sigma(x)\ \&\ A \subseteq B \Rightarrow B \in \sigma(x)$ (monotonicity)
- (c) $\Box A \wedge \Box B \to \Box(A \wedge B) \quad \leftrightsquigarrow \quad A \in \sigma(x)\ \&\ B \in \sigma(x) \Rightarrow A \cap B \in \sigma(x)$ (closure under intersection)
- (n) $\Box\top \quad \leftrightsquigarrow \quad \forall x \in W : W \in \sigma(x)$ (contains the unit)

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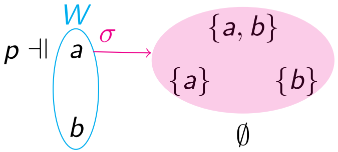

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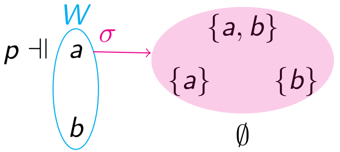

- $\mathfrak{M}$ is monotone, i.e., $\mathfrak{M} \Vdash$ (m)

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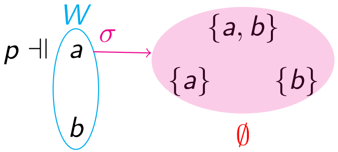

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- $\{a\} \cap \{b\} \notin \sigma(a)$, thus $\mathfrak{M} \nVdash$ (c)

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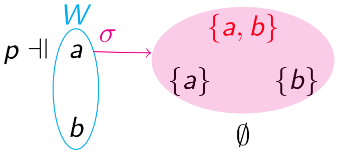

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- $W \notin \sigma(b)$, thus $\mathfrak{M} \nVdash$ (n).

A Sequent Calculus for Classical Modal Logic

We need a base calculus for logic E which we can extend with rules.

Definition

The sequent calculus sE contains the standard propositional rules and the modal sequent rule

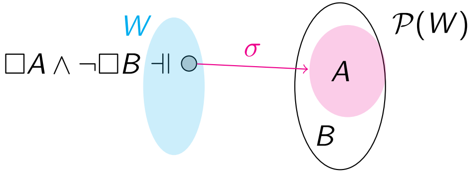
$$\frac{A \vdash B \quad B \vdash A}{\Gamma, \Box A \vdash \Box B, \Delta}\ \mathsf{Cg}$$

Theorem ([Lavendhomme, Lucas:'00])

sE *is sound and cut-free complete for* E*.*

Sketch of proof.

Soundness: From a countermodel $(W, \sigma, V), w$ for the conclusion we obtain one for a premiss:



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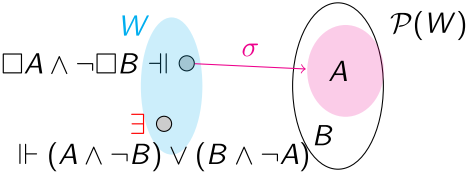

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Sketch of proof.

Completeness: simulate the Hilbert-system using cut and show cut elimination.

A Sequent Calculus for Classical Modal Logic

The cut elimination proof is essentially the standard one.
The only interesting case is:

$$\dfrac{\dfrac{A \vdash B \quad B \vdash A}{\Gamma, \Box A \vdash \Box B, \Delta}\,\text{Cg} \qquad \dfrac{B \vdash C \quad C \vdash B}{\Sigma, \Box B \vdash \Box C, \Pi}\,\text{Cg}}{\Gamma, \Sigma, \Box A \vdash \Box C, \Delta, \Pi}\,\text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{A \vdash B \quad B \vdash C}{A \vdash C}\,\text{cut} \qquad \dfrac{C \vdash B \quad B \vdash A}{C \vdash A}\,\text{cut}}{\Gamma, \Sigma, \Box A \vdash \Box C, \Delta, \Pi}\,\text{Cg}$$

□

How to construct calculi from modal axioms, then?

What about structural connectives and rules?

Interpreting the nesting of nested sequents with $\Box$ and using Ackermann's Lemma we have the following equivalences:

$$\frac{\Gamma, [A] \quad \Gamma, [B]}{\Gamma, [A \wedge B]} \quad \leftrightsquigarrow \quad \overline{\Box A \wedge \Box B \rightarrow \Box(A \wedge B)}$$

$$\overline{[\neg p, p]} \quad \leftrightsquigarrow \quad \overline{\Box(p \rightarrow p)}$$

$$\frac{\Gamma, [A]}{\Gamma, [A \vee B]} \quad \leftrightsquigarrow \quad \overline{\Box A \rightarrow \Box(A \vee B)}$$

Note that these are (equivalent over E to) the axioms (c), (n), (m). Thus (since MCN = K):

"Deep" admissibility of the propositional rules implies normality!

Hence the purely structural approach is problematic.
(But see, e.g., [Frittella:'14, L., Pimentel:'15])

Constructing sequent calculi from axioms

How do we construct calculi from modal axioms, then?

Alternative strategy: cut elimination by saturation

- Convert axioms to logical sequent rules.
  (The resulting system is usually not cut-free!)
- Massage (or saturate) the rules set so that it has cut elimination.

Since the initially constructed rules are not cut-free we need:

Key ingredients:

- A general cut elimination theorem specifying sufficient conditions.
- A general method for saturating rule sets so that they satisfy these conditions.

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Key ingredients:

- A general cut elimination theorem specifying sufficient conditions.
- A general method for saturating rule sets so that they satisfy these conditions.
- Bonus: A general decidability and complexity theorem.

Rank-1 axioms

We consider the ideas in a slightly simpler setting with axioms of a restricted form. (They can be generalised, of course.)

Definition

A rank-1 axiom is an axiom where every occurrence of a variable is under exactly one modality.

Examples

- The following axioms are rank-1 axioms:

  (m) $\Box(A \wedge B) \to \Box A$ (c) $\Box A \wedge \Box B \to \Box(A \wedge B)$ (n) $\Box\top$

- The reflexivity axiom $\Box A \to A$ is not a rank-1 axiom.
- The transitivity axiom $\Box A \to \Box\Box A$ is not a rank-1 axiom.

Rank-1 axioms

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Fact

Every rank-1 axiom is equivalent to a conjunction of rank-1 clauses of the form

$$\Box L_1 \wedge \cdots \wedge \Box L_n \to \Box R_1 \vee \cdots \vee \Box R_k$$

where the L_i and the R_j are purely propositional formulae.

Step 1: Axioms to Rules

To convert a rank-1 axiom, break it into rank-1 clauses.

Then, e.g., for the rank-1 clause

$$(c) \quad \frac{}{\vdash \Box A \wedge \Box B \to \Box(A \wedge B)}$$

- invert the propositional rules

$$\frac{}{\Box A, \Box B \vdash \Box(A \wedge B)}$$

- use the Tseitin transformation to replace propositional formulae under modalities with variables

$$\frac{A \vdash r \quad r \vdash A \quad B \vdash s \quad s \vdash B \quad A \wedge B \vdash t \quad t \vdash A \wedge B}{\Box r, \Box s \vdash \Box t}$$

- invert the propositional rules in the premisses

$$\frac{A \vdash r \quad r \vdash A \quad B \vdash s \quad s \vdash B \quad A, B \vdash t \quad t \vdash A \quad t \vdash B}{\Box r, \Box s \vdash \Box t}$$

- cut out superfluous formulae from the premisses (here: A, B)

$$\frac{r, s \vdash t \quad t \vdash r \quad t \vdash s}{\Box r, \Box s \vdash \Box t}\ \mathsf{C}$$

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Step 1: Axioms to Rules

To convert a rank-1 axiom, break it into rank-1 clauses.

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$$(\mathrm{n}) \quad \frac{}{\vdash \Box\top}$$

- invert the propositional rules

$$\frac{}{\vdash \Box\top}$$

- use the Tseitin transformation to replace propositional formulae under modalities with variables

$$\frac{\top \vdash r \quad r \vdash \top}{\vdash \Box r}$$

- invert the propositional rules in the premisses

$$\frac{\vdash r}{\vdash \Box r}$$

- cut out superfluous formulae from the premisses

$$\frac{\vdash r}{\vdash \Box r}\ \mathrm{N}$$

The crucial lemma for the cutting step

Lemma (Soundness of Cuts)

The rules below are interderivable in sEcut *(all p shown):*

$$\frac{\Omega \vdash \Theta, p \quad p, \Sigma_1 \vdash \Pi_1 \quad p, \Sigma_2 \vdash \Pi_2}{\Gamma \vdash \Delta} \qquad \frac{\Omega, \Sigma_1 \vdash \Theta, \Pi_1 \quad \Omega, \Sigma_2 \vdash \Theta, \Pi_2}{\Gamma \vdash \Delta}$$

Proof.

The tricky bit is to derive the premisses of the left rule from those of the right rule. For this we construct a formula for p and do:

$$\frac{\Omega, \Sigma_1 \vdash \Theta, \Pi_1 \quad \Omega, \Sigma_2 \vdash \Theta, \Pi_2}{\Omega \vdash \Theta, (\bigwedge \Sigma_1 \to \bigvee \Pi_1) \land (\bigwedge \Sigma_2 \to \bigvee \Pi_2)} \; prop$$

$$\frac{\overline{\bigwedge \Sigma_1 \to \bigvee \Pi_1, \Sigma_1 \vdash \Pi_1} \; prop}{(\bigwedge \Sigma_1 \to \bigvee \Pi_1) \land (\bigwedge \Sigma_2 \to \bigvee \Pi_2), \Sigma_1 \vdash \Pi_1} \; prop$$

□

Step 2: What about cut?

The crucial difference to the structural rules considered before:

The rule sets obtained from this procedure generally are not cut-free. E.g. we cannot reduce the cut

$$\dfrac{\dfrac{A, B \vdash C \quad C \vdash A \quad C \vdash B}{\Box A, \Box B \vdash \Box C}\,\mathsf{C} \qquad \dfrac{C, D \vdash E \quad E \vdash C \quad E \vdash D}{\Box C, \Box D \vdash \Box E}\,\mathsf{C}}{\Box A, \Box B, \Box D \vdash \Box E}\,\mathsf{cut}$$

The solution is to simply add the missing rule to the rule set:

$$\dfrac{A, B, D \vdash E \quad E \vdash A \quad E \vdash B \quad E \vdash D}{\Box A, \Box B, \Box D \vdash \Box E}$$

Note that the premisses of this rule are obtained by cutting superfluous formulae from the premisses of the derivation above (seen as a "macro rule").

The previous lemma ensures that this rule is sound.

Step 2: What about cut?

Definition

A modal rule set is saturated if it is closed under the addition of the missing rules from the previous slide and the rules required to meet the closure condition (closure under contraction).

Theorem (Cut elimination)

In a saturated rule set contraction and cut are admissible.

Proof.

The standard ones, with the interesting case:

$$\dfrac{\dfrac{\mathcal{P}_R}{\Gamma \vdash \Delta, \Box A}\,R \qquad \dfrac{\mathcal{P}_Q}{\Box A, \Sigma \vdash \Pi}\,Q}{\Gamma \vdash \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{P}_R \quad \mathcal{Q}_R}{(\mathcal{P}_R \cup \mathcal{P}_Q) \ominus A}}{\Gamma \vdash \Delta}\,\text{cut}(R, Q)$$

(Where $(\mathcal{P}_R \cup \mathcal{P}_Q) \ominus A$ comes from $\mathcal{P}_R \cup \mathcal{P}_Q$ by cutting on A in all possible ways.) □

Examples

Constructing cut-free calculi by this method starting from

$$\text{(c) } \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$$

for logic MC results first in the rules

$$\frac{A_1, \ldots, A_n \vdash B \quad B \vdash A_1 \quad \ldots \quad B \vdash A_n}{\Box A_1, \ldots, \Box A_n \vdash \Box B}\ \mathsf{C}_n$$

for $n \geq 1$. Adding (m) $\Box(A \wedge B) \rightarrow \Box A$ and saturating yields the rules

$$\frac{A_1, \ldots, A_n \vdash B}{\Box A_1, \ldots, \Box A_n \vdash \Box B}\ \mathsf{MC}_n$$

for logic MC. Finally, adding (n) $\Box\top$ gives the well-known rules

$$\frac{A_1, \ldots, A_n \vdash B}{\Box A_1, \ldots, \Box A_n \vdash \Box B}\ \mathsf{K}_n$$

($n \geq 0$) for logic MCN, i.e., modal logic K!

Bonus: Decidability and complexity

So, what can we do with the calculi?

Theorem

Derivability in a saturated rule set is decidable in polynomial space.

Proof.

By the standard backwards proof search algorithm:

On input $\Gamma \vdash \Delta$:

- if $\Gamma \vdash \Delta$ is initial sequent, then accept; otherwise
- existentially guess a rule with conclusion $\Gamma \vdash \Delta$
- universally choose a premiss $\Sigma \vdash \Pi$ of this rule
- recursively call the algorithm with input $\Sigma \vdash \Pi$.

The complexity of the sequents strictly decreases from conclusion to premisses in every rule, so branches of the search tree have *polynomial length*. By complexity theory we get PSPACE. □

Extensions: Nested axioms and hypersequents

Cut elimination by saturation extends from rank-1 axioms to:

The translation to rules applies to axioms given by:

$$
\begin{aligned}
S_{\text{init}} &::= \quad L \to R \\
L &::= L \wedge L \mid \Box P_r \mid \top \mid \bot \qquad\qquad R ::= R \vee R \mid \Box P_\ell \mid \top \mid \bot \\
P_r &::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \qquad \mid p_i \mid \bot \mid \top \\
P_\ell &::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \qquad \mid p_i \mid \bot \mid \top
\end{aligned}
$$

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- Shallow axioms with propositional variables on the top-level, e.g., (t) $\Box A \to A$

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$$L ::= L \wedge L \mid \Box P_r \mid \top \mid \bot \mid p_i \qquad R ::= R \vee R \mid \Box P_\ell \mid \top \mid \bot \mid p_i$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \qquad \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \qquad \mid p_i \mid \bot \mid \top$$

Extensions: Nested axioms and hypersequents

Cut elimination by saturation extends from rank-1 axioms to:

- Shallow axioms with propositional variables on the top-level, e.g., (t) $\Box A \to A$
- Axioms with a limited form of nested modal operators, e.g., (4) $\Box A \to \Box\Box A$

The translation to rules applies to axioms given by:

$$S_{\text{init}} ::= \quad L \to R$$

$$L ::= L \wedge L \mid \Box P_r \mid \top \mid \bot \mid p_i \mid \psi_\ell \qquad R ::= R \vee R \mid \Box P_\ell \mid \top \mid \bot \mid p_i \mid$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\psi_\ell \in \{q_i, \Box q_i : i \in \mathbb{N}\}$, $\psi_r \in \{r_i : i \in \mathbb{N}\}$ such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality.

Extensions: Nested axioms and hypersequents

Cut elimination by saturation extends from rank-1 axioms to:

- Shallow axioms with propositional variables on the top-level, e.g., (t) $\Box A \to A$
- Axioms with a limited form of nested modal operators, e.g., (4) $\Box A \to \Box\Box A$
- The hypersequent framework, e.g., (5) $\Box A \vee \Box\neg\Box\neg A$.

The translation to rules applies to axioms given by:

$$S_{\text{init}} ::= \Box(L \to R) \vee \cdots \vee \Box(L \to R)$$

$$L ::= L \wedge L \mid \Box P_r \mid \top \mid \bot \mid p_i \mid \psi_\ell \qquad R ::= R \vee R \mid \Box P_\ell \mid \top \mid \bot \mid p_i \mid$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\psi_\ell \in \{q_i, \Box q_i : i \in \mathbb{N}\}$, $\psi_r \in \{r_i : i \in \mathbb{N}\}$ such that every ψ_ℓ, ψ_r occurs in the $L \to R$ once on the top level and at least once under a modality.

Extensions: Nested axioms and hypersequents

Of course, there is also a price to pay:

- Naive saturation yields infinitely many rules – concise presentations need to be found manually (yet...)
- From nested modal operators onwards, simple saturation of the rule set is not sufficient for cut elimination anymore

Example

Converting the K-axioms as well as (t) $\Box A \to A$, (4) $\Box A \to \Box\Box A$ and (5) $\Box A \vee \Box\neg\Box\neg A$ to rules and saturating yields the rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \qquad \frac{\Box\Gamma, \Sigma \vdash A, \Box\Delta}{\Box\Gamma, \Box\Sigma \vdash \Box A, \Box\Delta}$$

In Lecture 2 we saw that this rule set is not cut-free.

Evaluation: Structural versus Logical Approaches

	Structural rules	Logical rules
Pros:	✓ Elegant and clean rules	✓ No additional structural connectives
	✓ Cut elimination for free	✓ capture non-normal modal logics
	✓ Modular calculi	✓ good for proof search
Cons:	✗ need additional structural connectives	✗ usually infinitely many rules
	✗ higher complexity for proof search	✗ cut elimination not automatic
	✗ difficult to handle non-normal modalities	

An IndoLogical problem revisited.

Constructing a Mīmāṃsā deontic logic

With these tools our indologist now can approach her problem.

A promising language might include

- a modality $\Box$ to model necessity
- a binary modality $\mathcal{O}(\cdot/\cdot)$ to model conditional obligation: a formula

$$\mathcal{O}(A/B)$$

  reads “under the conditions B it is obligatory that A”.

(The methods above extend readily to this.)

As a starting point we take $\Box$ to be a S4-modality with the axioms

$$(\mathrm{t}) \quad \Box A \to A \qquad (4) \quad \Box A \to \Box\Box A$$

Constructing a Mīmāṃsā deontic logic

The principle

यत्र तूत्पत्त्यादयो न विध्यन्तरसिद्धास् तत्र स्वयमेव
स्वसम्बन्धिनामुत्पत्त्यादिचतुष्टयं करोति

(I.e., "When, on the other hand, coming into existence [of something needed], etc., are not realised by another prescription, [the principal prescription] itself begets the four [stages] of coming into being, etc., [of the prescriptions] connected to itself.")

and two other principles could be formalised as the axioms

$$\Box(A \to B) \to (\mathcal{O}(A/C) \to \mathcal{O}(B/C))$$

$$\Box(B \to \neg A) \to \neg(\mathcal{O}(A/C) \wedge \mathcal{O}(B/C))$$

$$\Box(B \leftrightarrow C) \wedge \mathcal{O}(A/B) \to \mathcal{O}(A/C)$$

Constructing a Mīmāṃsā deontic logic

Conversion into rules and saturation with the standard S4-rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{T} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ 4$$

gives the rules

$$\frac{\Box\Gamma, A \vdash C \quad \Box\Gamma, B \vdash D \quad \Box\Gamma, D \vdash B}{\Box\Gamma, \mathcal{O}(A/B) \vdash \mathcal{O}(C/D)}\ \mathsf{Mon}$$

$$\frac{\Box\Gamma, A \vdash}{\Box\Gamma, \mathcal{O}(A/B) \vdash}\ \mathsf{D}_1 \qquad \frac{\Box\Gamma, A, C \vdash \quad \Box\Gamma, B \vdash D \quad \Box\Gamma, D \vdash B}{\Box\Gamma, \mathcal{O}(A/B), \mathcal{O}(C/D) \vdash}\ \mathsf{D}_2$$

Theorem

The calculus with the above modal rules has cut elimination and derivability is decidable in exponential time.

A Mīmāṃsā deontic logic

The question now might arise whether this is "the right" logic.

Minimal requirement: consistency with seemingly contradictory statements from the vedas, e.g., the problem of the Śyena:

- *You should not harm any living being*
- *If you desire to harm your enemy, you should perform the Śyena sacrifice*

The statement that this is contradictory could be formalised as

$$\Box(\mathrm{hrm\_e} \to \mathrm{hrm}), \Box(\mathrm{sy} \to \mathrm{hrm\_e}), \Box\mathcal{O}(\neg\mathrm{hrm}/\top), \Box\mathcal{O}(\mathrm{sy}/\mathrm{des\_hrm}) \vdash \bot$$

Backwards proof search gives:

Theorem

The problem of the Śyena is not contradictory in Mīmāṃsā deontic logic, i.e., the above sequent is not derivable.

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